6.302 Design Project

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I used data for oven number 11.

Results Summary

Problem	Part	Answer
1	a	$\omega_c = 1$
1	b	$\phi_M = 45°$
1	c	$g_M = \infty$
1	d	$e_{ss}(\infty) = 0$
1	e	$e_{ss}(\infty) = 0.707$
1	f	$P_o = 1.23$
1	g	Noise attenuated by a factor of 7071
1	h	$t_r = 1.36$
1	i	$t_s = 14.03$
1	j	$\omega_h = 1.61$
2	a	$K_R = 0.134$
2	b	$\tau = 0.0909$, $\alpha = 17.37$
2	c	$\tau = 3.47 \times 10^{-4}$, $K_L = 0.434$
3	2 (a)	$K_R = 0.412$
3	2 (b)	$K_P = 0.0339$
3	2 (c)	$\tau = 1.26$, $\alpha = 2.95$
3	2 (d)	$\tau = 0.0251$, $K_L = 0.211$

Problem	Reduced Gain	Dominant Pole	Lag	Lead
3.3 (a)	0.794	0	0.333	1.58
3.3 (b)	∞	9.833	∞	∞
3.3 (c)	1.27	1.27	1.27	1.27
3.3 (d)	1.6	1977	1.98	0.342
3.3 (e)	0.199	2.32	0.232	0.146
3.3 (f)	2.30	26.8	13.1	1.69

Note that for Problem 3, part 3 (d), the numbers are the factors by which noise is attenuated. In other words, the noise transfer function at that frequency is one divided by the number in the table. I used data for oven number 11.

Problem 1

(a) $\omega_c = 1$

(b) $\phi_M = 45°$

(c) $g_M = \infty$ since $\angle L(j\omega)$ approaches $-180°$ in the limit as ω grows without bound, but never actually reaches $-180°$ for any finite frequency.

(d) Since the error transfer function for a unity feedback system is $\frac{E}{R} = \frac{1}{1+L(s)}$, the final value theorem says that the steady state error for a unit step input should be:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} s\frac{E}{R}\frac{1}{s} = \lim_{s\to 0} \frac{s(s+1)}{s(s+1)+\sqrt{2}} = 0$$

(e)

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} s\frac{E}{R}\frac{1}{s^2} = \lim_{s\to 0} \frac{s+1}{s(s+1)+\sqrt{2}} = \frac{1}{\sqrt{2}} = 0.707$$

(f) Since the closed loop transfer function is $\frac{C}{R} = \frac{\sqrt{2}}{s^2+s+\sqrt{2}}$, the natural frequency is $\omega_n = \frac{1}{\sqrt{\sqrt{2}}} = 1.189$ and the damping ratio is $\zeta = \frac{1}{2\sqrt{\sqrt{2}}} = 0.420$. From these parameters, we can calculate the peak overshoot using the relation:

$$P_o = 1 + e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} = 1.23$$

(g) For a disturbance injected in the feedback path of the system, the disturbance transfer function is $\frac{C}{D} = \frac{-L(s)}{1-L(s)}$. Since $\omega = 100$ is well above the crossover frequency, the magnitude of the loop gain will be quite small at that frequency and $|\frac{C}{D}| \approx |-L(j\omega)| = |G_p(j100)||G_c(j100)|$. Since $|L(j100)| = \frac{\sqrt{2}}{10^4}$, noise should be attenuated by a factor of 7071.

(h) Because this is a second order system, we can estimate the rise time using the closed loop bandwidth, which we can easily derive from the damping coefficient:

$$t_r \approx \frac{2.2}{\omega_h} = 1.36$$

We use the bandwidth value calculated in part j.

(i) For most second order systems, the settling time can be estimated by finding the time needed for the envelope to settle to within 0.1% of its final value:

$$\frac{e^{-t_s\zeta\omega_n}}{\sqrt{1-\zeta^2}} = 0.001$$

$$t_s = \frac{-\log(0.001\sqrt{1-\zeta^2})}{\zeta\omega_n} = 14.03$$

(j) The bandwidth is the frequency at which the magnitude of the closed loop system equals $\frac{1}{\sqrt{2}}$. Since we know the transfer function of the closed loop system and we know that its DC gain is one, we calculate the bandwidth like so:

$$|\frac{C}{R}| = \frac{\sqrt{2}}{\sqrt{\omega_h^2 + (\sqrt{2} - \omega_h^2)^2}} = \frac{1}{\sqrt{2}}$$

Solving for ω_h, we discover that $\omega_h = 1.61$

Problem 2

In order to minimize steady state error, we need high open loop DC gain. In order to maximize closed loop bandwidth, we need a high open loop crossover frequency (see Problem 3 part 3.e for an explanation of why). In general, we optimize the series compensator by picking the highest crossover point at which we can meet our phase margin requirement without excessively reducing DC gain.

(a) Reduced Gain Compensator

The highest frequency at which $\angle G(j\omega_c) = -180° + 30°$ is $\omega_c = 193$ so we make this our new crossover frequency. In order to do so, we must reduced the overall loop gain by exactly the magnitude of the plant at that frequency:

$$K_R = \frac{1}{|G_p(j\omega_c)|} = \frac{1}{7.455} = 0.134$$

(b) Lag Compensator

The new crossover should occur at the highest frequency at which $\angle G(j\omega_c) = -144°$, which gives us $\omega_c = 110$. In order to minimize the negative phase bounce associated with lag at crossover, we want to place the lag zero far below crossover. In order to reduce error and improve transient performance, we want to place the lag zero close to crossover. We compromise by placing it one decade below crossover giving $\frac{1}{\tau} = \frac{\omega_c}{10}$ and hence $\tau = \frac{1}{11}$ seconds. In order to crossover at the desired frequency, the compensator magnitude at that frequency must be the inverse of the plant magnitude at that frequency:

$$|G_c(j\omega_c)| = \frac{1}{|G_p(j\omega_c)|} = \frac{1}{17.37}$$

$$|\frac{\tau s + 1}{\alpha\tau s + 1}| = \frac{1}{17.37} \approx \frac{1}{\alpha}$$

Therefore, $\alpha = 17.37$.

(c) Lead Compensator

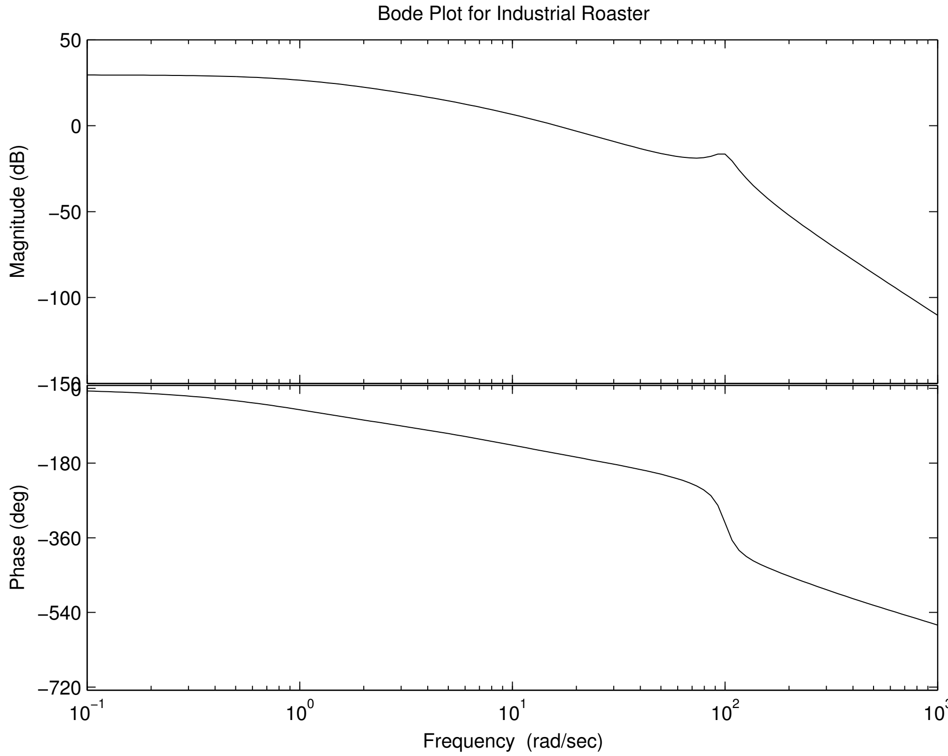
Because we have a preselected $\alpha = 10$, we know that the maximum phase bump associated with this lead compensator will be $sin^{-1}\frac{9}{11} = 54.9°$. The new crossover frequency should occur where $\angle G(j\omega_c) = -205°$, which gives us $\omega_c = 910$. Since the lead must be centered at the new crossover frequency for maximal benefit, we select $\tau = \frac{1}{\omega_c\sqrt{10}} = 3.47 \times 10^{-4}$. We choose K_L by picking the value needed to make compensated system have a magnitude of one at the new crossover frequency:

$$K_L|G_p(j\omega_c)|\frac{\sqrt{1 + 100\tau^2\omega_c^2}}{\sqrt{1 + \tau^2\omega_c^2}} = K_L 0.7282\sqrt{10}$$

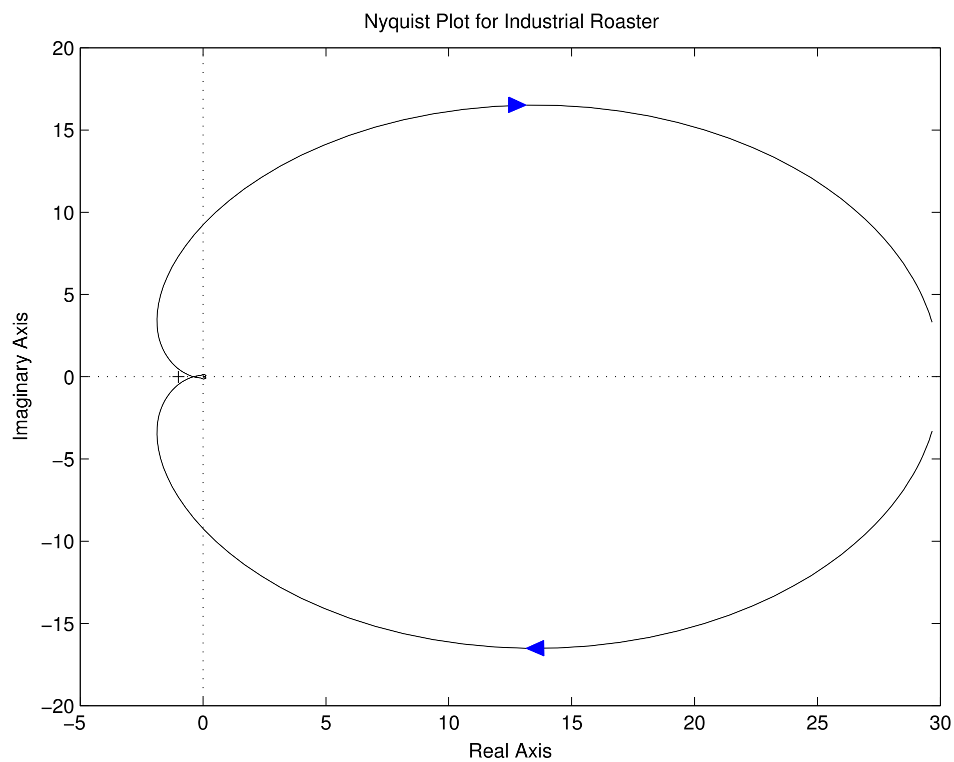
Therefore, $K_L = 0.434$.

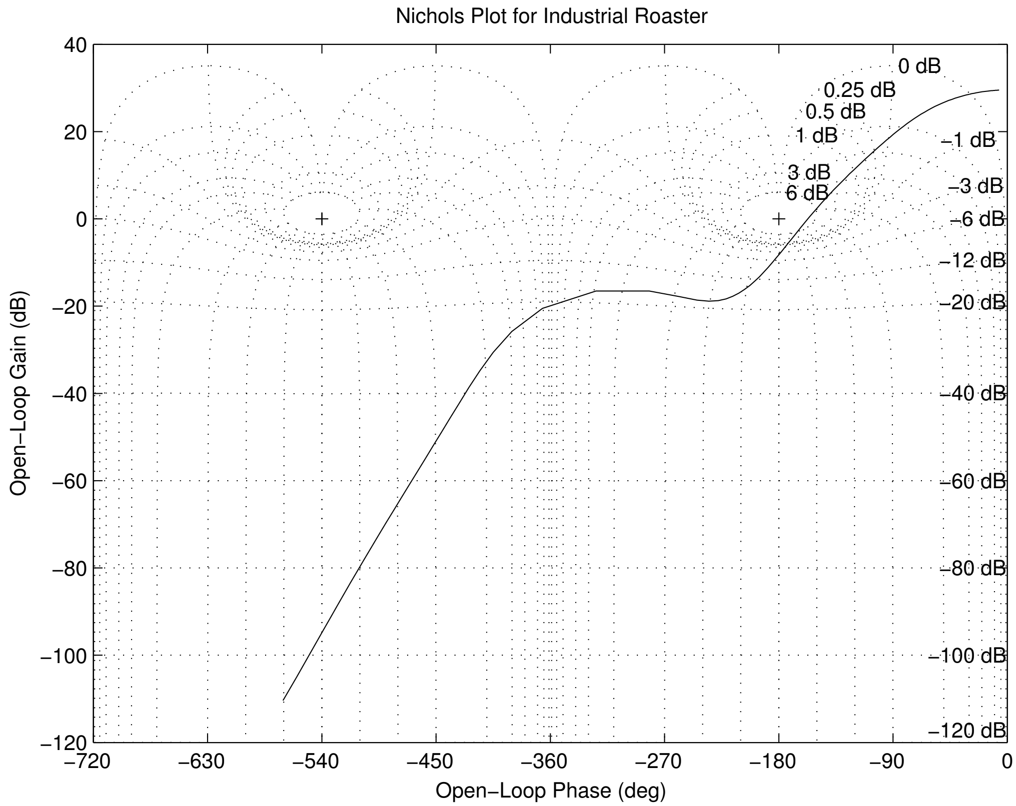
Problem 3

1. (a)



(b)





(c)

2. (a) Reduced Gain Compensator

To achieve a phase margin of $\phi_m = 45$ while maximizing closed loop bandwidth and open loop DC gain, we need force the open loop system to cross over at the frequency where $\angle L(j\omega_c) = -135°$. This occurs at $\omega_c = 9.261$. In order to force crossover to occur at that frequency, we set our compensator to be the inverse of the magnitude of the open loop gain at that frequency:

$$K_R = \frac{1}{|G_p(j\omega_c)|} = \frac{1}{2.383} = 0.4196$$

(b) Dominant Pole Compensator

Since the compensator integrator adds $-90°$ of phase, closed loop bandwidth and open loop DC gain will be maximized if we force crossover to occur at the frequency where $\angle L(j\omega_c) = -45°$. This occurs at $\omega_c = 0.7943$. We choose K_P by making the loop gain have a magnitude of one at the new crossover frequency:

$$|L(j\omega_c)| = \frac{K_P}{\omega_c}|G_p(j\omega_c)| = 1$$

$$K_P = \frac{\omega_c}{|G_p(j\omega_c)|} = 0.0339$$

(c) Lag Compensator

In order to maximize open loop DC gain and closed loop bandwidth, we place the lag as high as possible but sufficiently below the new crossover point so as not excessively reduce phase margin. We compromise by placing the lag zero one decade below crossover, ensuring that the lag only contributes -6° of phase shift at crossover. We then choose the crossover frequency as the lowest frequency for which $\angle L(j\omega_c) = -180^\circ + 45^\circ + 6^\circ = -129^\circ$. This occurs at $\omega_c = 7.943$. Since we've decided to place the lag zero one decade below crossover, $\frac{1}{\tau} = \frac{\omega_c}{10}$ and $\tau = 1.26$. To select the location of the pole, we solve for the value of alpha needed to force the magnitude of the loop gain equal one at the new crossover frequency:

$$|L(j\omega_c)| = |\frac{\tau s + 1}{\alpha \tau s + 1}||G_p(j\omega_c)| = 1$$

$$1 = 2.95\frac{\sqrt{1 + 100}}{\sqrt{1 + 100\alpha^2}}$$

$$\alpha = 2.95$$

(d) Lead Compensator

Ordinarily we would place the lead compensator by choosing a new crossover frequency where $\angle L(j\omega_c) = -189^\circ$. However, because of magnitude peaking in the plant system function, the resulting system is unstable (or has unacceptably small gain margin with appropriate choices of K_L). In order to provide adequate gain margin, we choose the new crossover frequency to be the highest frequency below the peak that has a magnitude 20 dB above the magnitude of the plant peak. This happens at $\omega_c = 12.6$. In order to center the lead at the new crossover frequency, we choose $\tau = \frac{1}{\omega_c\sqrt{10}} = 0.0251$. Finally, in order to make the compensated system actually crossover at ω_c, we set $K_L = \frac{1}{|G_p(j\omega_c)|\sqrt{10}} = 0.211$.

3. (a) Steady State Error for a unit step input

Since this is a unity feedback system, the error transfer function is $\frac{E}{R} = \frac{1}{1+L(s)}$. For very low frequencies, $|L(j\omega)| \gg 1$ so $\frac{E}{R} \approx \frac{1}{L(s)}$. The steady state error for a unit step is thus approximated by:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)\frac{1}{s} \approx \lim_{s\to 0} \frac{1}{G_c(s)G_p(s)}$$

For the reduced gain system, steady state error is approximately $\frac{1}{K_R|G_p(j0)|} = 0.794$.

For the dominant pole compensated system, steady state error is zero.

For the lag compensated system, steady state error is approximately $\frac{1}{|G_p(j0)|} = 0.333$.

For the lead compensated system, steady state error is approximately $\frac{1}{K_L|G_p(j0)|} = 1.58$.

(b) Steady State Error for a unit ramp input

Using the same logic as above, we can see that for the reduced gain, lag, and lead compensated systems, the steady steady error to a unit ramp is infinite. For the dominant pole compensated system, the error is $\frac{1}{K_P|G_p(j0)|} = 9.833$.

(c) Step Response Peak Overshoot

Since all of the compensated systems have a phase margin of 45°, we can estimate the equivalent magnitude peaking M_p. Once we know M_p, we can derive the damping coefficient ζ. Given ζ, we can easily calculate the peak overshoot:

$$M_p \approx \frac{1}{\sin\phi_M} = 1.414$$

$$\zeta = \sqrt{\frac{1 \pm \sqrt{1 - \frac{1}{M_p^2}}}{2}} = 0.383$$

$$P_o = 1 + e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} = 1.27$$

All the compensated systems have an estimated peak overshoot of 1.27.

(d) Noise rejection at $\omega = 100$

For a disturbance injected in the feedback path of the system, the disturbance transfer function is $\frac{C}{D} = \frac{-L(s)}{1-L(s)}$. Since $\omega = 100$ is well above the crossover frequency of any of the compensated systems, the magnitude of the loop gain will be quite small at that frequency and $|\frac{C}{D}| \approx |-L(j\omega)| = |G_p(j\omega)||G_c(j\omega)|$. For the reduced gain compensator, noise will be attenuated by a factor of 1.6. For the dominant pole compensator, noise will be attenuated by a factor of 1977. For the lag compensator, noise will be attenuated by a factor of 1.98. For the lead compensator, noise will be attenuated by a factor of 0.342; in other words, noise will be amplified by a factor of 2.93 in the output.

(e) Step Response Rise Time

Since these are approximately second order systems, we can say that $t_r \approx \frac{2.2}{\omega_h}$. In order to calculate the closed loop bandwidth, we rely on the fact that all the compensated systems have the same phase margin and hence, the same magnitude peaking in the frequency domain. Because of that, the ratio $\frac{\omega_h}{\omega_c}$ is constant for all these systems. Using the value of $\frac{\omega_h}{\omega_c} = 1.193$ calculated from problem 1, we can determine the closed loop bandwidth for each of the compensated systems.

The reduced gain system has a $t_r = 0.199$ seconds. The dominant pole compensated system has a $t_r = 2.32$ seconds. The lag compensated system has a $t_r = 0.232$ seconds. The lead compensated system has a $t_r = 0.146$ seconds.

(f) Step Response Settling Time (0.1%)

Since the reduced gain, dominant pole, and lead compensated systems are dominated by second order effects, we can approximate the 0.1% settling time $t_s \approx \frac{6.9}{\zeta\omega_n}$. We can estimate ω_n by using our estimate of ω_h and ζ. For the reduced gain system, we expect a settling time of 2.30 seconds. For the dominant pole compensated system, we expect a settling time of 26.8 seconds. For the lead compensated system, we expect a settling time of 1.69 seconds.

Because the lag compensator introduces a low frequency zero and pole pair near each other, the lag compensated system has the long settling time associated with a doublet. We estimate the settling time of the lag compensated system by calculating the settling time of the lag compensator by itself. The lag compensator should have a time constant of 1.9 seconds by itself; That system settles to within 0.1% of its final value when $t_s = -1.9 \log 0.001 = 13.1$ seconds. Therefore, the lag compensated system should have a settling time of about 13.1 seconds.

4.